The Simple Analytics of a Pooled Annuity Fund*

by John Piggott (corresponding author)
Emiliano A. Valdez
Bettina Detzel

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Abstract

This paper provides a formal analysis of payout adjustments from a longevity-risk-pooling fund, an arrangement we refer to as group self annuitization (GSA). The distinguishing risk diffusion characteristic of GSAs in the family of longevity-insurance instruments is that the annuitants bear their systematic risk, but the pool shares idiosyncratic risk. This obviates the need for an insurance company, although such instruments could be sold through a corporate insurer. We begin by deriving the payout adjustment for a single-entry group with a single-annuity factor and constant expectations. We then show that, under weak requirements, a unique solution to payout paths exists when multiple cohorts combine into a single pool. This relies on the harmonic mean of the ratio of realized-to-expected survivorship rates across cohorts. The case of evolving expectations is also analyzed. In all cases, we demonstrate that the periodic benefit payment in a pooled annuity fund is determined based on the previous payment adjusted for any deviations in mortality and interest from expectations. GSA may have considerable appeal in countries which have adopted national defined-contribution schemes and/or in which the life insurance industry is noncompetitive or poorly developed.

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1. Introduction and Motivation

From a theoretical perspective, annuitization is a natural mechanism for insuring against longevity risk, especially at retirement. Risk-averse individuals value annuities highly (Mitchell, 2002). However, voluntary annuity markets remain thin, and there is evidence that risk-sharing through transfers is limited even within families (Hayashi, et al., 1996). Annuity demand remains low despite tax concessions, perhaps because annuity loadings are often penal, especially in small economies such as Australia's (see, for example, Doyle, et al., 2001), or in countries where the financial sector is not well developed. Supply also appears reluctant, perhaps because of an industry perception that systematic risk, in the form of breakthrough life-prolonging technical innovation, may bankrupt an insurance company with a large life-annuity portfolio. This has led to a situation where almost no voluntary longevity risk-spreading takes place in the private market, in spite of its clear welfare-enhancing effects (Kotlikoff and Spivak, 1981; Kingston and Piggott, 1999).

Milevsky and Robinson (2001) develop the mathematical analysis that can be made by individuals at retirement who face the choice between voluntary annuitization and self-annuitization. Self-annuitization provides greater liquidity than voluntary annuitization; however it does so at the cost of possibly outliving resources. Albrecht and Maurer (2002) evaluate that risk by calculating a personal probability of consumption shortfall and show that it is substantial, particularly for high-entry ages.

A possible response is to separate the systematic from the idiosyncratic risk. Groups could be formed to pool idiosyncratic risk within a clear framework with specified legal rights and obligations, but payouts could be conditioned to the mortality experience of the group. This concept of GSA was mooted by Wadsworth et al. (2001) and Martineau (2001). A group-self-annuity plan will allow retirees to pool together and form a fund that can provide for protection against longevity. With the right implementation, GSA can provide a less expensive form of insurance against the risk of longevity.

This is not the only contribution in recent times to address the problem of systematic longevity risk. An alternative policy strategy which would allow annuity issuers to immunize their systematic longevity risk is for the government to issue "survivor bonds," in which payouts are linked to evolving mortality in a manner analogous to "indexed bonds," whose payouts are linked to evolving price inflation (Blake and Burrows, 2001). Thus far, no such bonds have been issued.

In the United States, annuities where payments reflect evolving mortality have for some time been issued by the Teachers Insurance and Annuity Association (TIAA) through its companion organization College Retirement

Equities Fund (CREF). Mortality experience of its participants receiving lifetime income is tracked and this historical experience is used as a guide in the annual adjustment to the mortality participation factor. TIAA-CREF reports that historically, the impact of annual mortality adjustments has been relatively modest.[1] Brown et al. (2001) provide additional commentary on the TIAA-CREF product. Weil and Fischer (1974) indicate that in general, the initial benefit payment calculation is based on a conservative interest rate and mortality assumptions. Most of the adjustments have reflected investment experience and very little on errors resulting from mortality projections. However, there appears to be no formal treatment of how benefit adjustments for these types of products should be calculated, even with annuities issued by TIAA-CREF, or what conditions these adjustments should satisfy.

This paper aims to present a systematic procedure for valuing GSA payouts in the presence of changing mortality. It begins in section 2 by identifying the determinants of the periodic payment time path of a GSA payout, assuming that a single cohort with identical *a priori* mortality characteristics is participating, that there is a single safe asset with a constant rate of return and that there is no government intervention. This can be readily generalized to the case where the payouts are underwritten by a risky portfolio, and where members of the pool bring different accumulations for annuitization (section 3). We show that the ratio of the expected-to-actual proportion of survivors is central to these adjustment formulae.

More complicated cases are then introduced. The pool is "opened," so that successive cohorts (with differentiated annuity factors) may participate in the pool. In section 4 we extend our analysis to encompass multiple cohorts, joining the pool at arbitrary time points, with differentiated annuity factors. We find that under remarkably weak requirements, essentially equi-proportional payout adjustment, the adjustment is uniquely given by the harmonic mean of the expected-to-realized survival proportions for each cohort. In section 5, we allow expectations about mortality change to evolve over time as new information about mortality emerges. Section 6 provides a numerical illustration demonstrating the ideas developed in the paper. We conclude in section 7.

2. A Simple Actuarial Analysis of GSA Plans

A GSA plan will initially operate like an ordinary life annuity purchased in the private market so that much of the initial pricing procedure consists of calculating the annuity payout rate. This benefit payout formula must capture both the annuitant's expected mortality in the future, accounting for anticipated

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[1] All TIAA-CREF pension annuities have been priced on a merged-gender basis since 1983. Improvements in mortality in the United States for some time have tended to favor men.

mortality improvements, and the expected rate of return on the investment portfolio; for simplicity, we assume a flat yield curve. If these expectations are actually realized over time, the payout rates determined at the point of entry will remain constant. Assume that at time 0, a pool of ℓ_x annuitants, all aged x, decides on the amount they expect to receive periodically in the future. Suppose that amount is a level payment of B_0 so that the starting total fund is:

$$F_0 = \ell_x B_0 \sum_{t=0}^{\infty} \ell_{x+t} / \ell_x \cdot v^t = \ell_x B_0 \ddot{a}_x \tag{1}$$

where ℓ_z denotes the expected number of lives to survive to age z, $v = 1/(1+R)$ is the discount factor, and $\ddot{a}_x$ is a standard actuarial notation for the annuity factor. We use this notation throughout the paper; for convenience, Appendix B lists some of the actuarial and algebraic symbols we use. Its value is given by:

$$\ddot{a}_x = \sum_{t=0}^{\infty} v^t \cdot \ell_{x+t} / \ell_x ,$$

and is interpreted as the expected present value of a life annuity-due that pays a periodic payment of one at the beginning of the period.[2]

We now derive a principle for the development of future benefit payments in the case where the actual survival pattern is different from expected, i.e., the number of individuals in the fund surviving is different from expected. In this case, it is not possible to continue to pay out the level payment determined at the beginning since this would result in fund imbalance. The number of actual survivors will be superscripted by a * so that the actual number of survivors at each future period will be:

$$\ell^*_{x+1}, \ell^*_{x+2}, \ldots, \ell^*_{x+t}, \ldots$$

where ℓ_x is fixed and known. We assume that investment earnings rates will be realized as assumed. We now determine the distribution formula for future benefit payments (again superscripted by *). There remains no change at time 0 so that the payment per survivor is B_0 as in equation (1). At time 1, however, the fund becomes:

$$F_1 = (F_0 - \ell_x B_0)(1+R) = \ell_x B_0 (\ddot{a}_x - 1)(1+R).$$

Spreading this across the remaining survivors during their expected future lifetime, the periodic benefit payment becomes

[2] A similar formula can be derived for the case of an annuity-immediate.

$$B_1^* = \frac{1}{\ell^*_{x+1}}\left(\frac{F_1}{\ddot{a}_{x+1}}\right) = \frac{1}{\ell^*_{x+1}}\left(\frac{\ell_x B_0(\ddot{a}_x - 1)(1+R)}{\ddot{a}_{x+1}}\right).$$

Using the recursive relationship for annuity factors (see, for example, Bowers, et al., 1997),

$$\ddot{a}_{x+1} = (\ddot{a}_x - 1)(1+R)\cdot 1/p_x \tag{2}$$

where $p_x = \ell_{x+1}/\ell_x$, we have:

$$B_1^* = \frac{1}{\ell^*_{x+1}}\left(\frac{\ell_x B_0(\ddot{a}_x - 1)(1+R)}{(\ddot{a}_x - 1)(1+R)1/p_x}\right) = B_0\left(\frac{\ell_x p_x}{\ell^*_{x+1}}\right) = B_0\left(\frac{p_x}{p^*_x}\right)$$

where p denotes the expected annual survivorship rate, with the superscripted * denoting the realized annual survivorship rate. Note that the adjustment factor is based on the ratios of survivorship rates rather than the numbers of survivors.

Proceeding inductively, at any time t in the future, we would have the benefit payment determined as:

$$B_t^* = \frac{F_t^*}{\ell^*_{x+t}\ddot{a}_{x+t}} = \frac{\ell^*_{x+t-1}B^*_{t-1}(\ddot{a}_{x+t-1} - 1)(1+R)}{\ell^*_{x+t}\ddot{a}_{x+t}}$$
$$= B^*_{t-1}\frac{\ell^*_{x+t-1}}{\ell^*_{x+t}}\frac{(\ddot{a}_{x+t-1} - 1)(1+R)}{(\ddot{a}_{x+t-1} - 1)(1+R)\cdot 1/p_{x+t-1}} = B^*_{t-1}\left(\frac{p_{x+t-1}}{p^*_{x+t-1}}\right).$$

The extension to the case where the investment earnings pattern is different from the assumed constant rate of R is straightforward. Assume that the actual investment earnings rates are:

$$R_1^*, R_2^*, \ldots, R_t^*, \ldots$$

where the subscript denotes the period. At time t, the fund will equal to:

$$F_t^* = \left(F^*_{t-1} - \ell^*_{x+t-1}B^*_{t-1}\right)\left(1+R_t^*\right) = \ell^*_{x+t-1}B^*_{t-1}(\ddot{a}_{x+t-1} - 1)\left(1+R_t^*\right)$$

and spreading this across the remaining lives, we have:

$$B_t^* = \frac{F_t^*}{\ell^*_{x+t}\ddot{a}_{x+t}} = \frac{\ell^*_{x+t-1}B^*_{t-1}(\ddot{a}_{x+t-1} - 1)\left(1+R_t^*\right)}{\ell^*_{x+t}\ddot{a}_{x+t}}.$$

Using (2) and with some transformations, we have

$$B_t^* = B^*_{t-1}\frac{\ell^*_{x+t-1}}{\ell^*_{x+t}}\frac{(\ddot{a}_{x+t-1} - 1)}{(\ddot{a}_{x+t-1} - 1)(1+R)\cdot 1/p_{x+t-1}}\left(1+R_t^*\right) = B^*_{t-1}\left(\frac{p_{x+t-1}}{p^*_{x+t-1}}\times\frac{1+R_t^*}{1+R}\right). \tag{3}$$

Hence, we observe that the payment for period t depends on the payment for period $t-1$ and two adjustment factors: the first one is related to the difference in expected and realized mortality during the previous period and the second factor is related to the difference in the expected and realized investment earnings rate for the period.

The essential feature in the calculations demonstrated above, i.e., the result in formula (3), is that the periodic, here assumed annual, benefit payout rates can be determined from the previous benefit payout rates multiplied by two adjustment factors. The generic adjustment is given by

$$B_t^* = B_{t-1}^* \times MEA_t \times IRA_t \tag{4}$$

where MEA_t is the mortality experience adjustment and IRA_t is the interest rate adjustment for the period from year $t-1$ to t.

This is how a GSA plan is anticipated to operate: recompute the benefit payouts periodically using the most recent benefit payouts and multiply by adjustment factors. If, for example, mortality is lighter than expected for the period, it will lower the next period's benefit payouts. The intuition here is that the funds that accumulate will have to be spread across a larger surviving group and there is less "inheritance" than expected. Similarly, if investment earnings for the period were worse than expected, there will also be lower benefit payouts.

In the following sections, we extend formula (4) to include more complicated but realistic situations.

3. Varying Contributions and Annuity Payouts

The previous section developed a straightforward calculation of the benefit payout rates assuming that participants contribute equal amounts into the fund and in return, receive equal amounts of annuity benefit payments. Consider the case where we allow varying amounts of contributions and annuity payout rates for the participants. To fix notation, assume that at the beginning of the period $t=0$, there is a cohort A_0 of individuals all aged x who join the group. The *i*-th annuitant brings an amount $F_{i,0}$ into the fund at $t=0$ so that the total fund at the beginning of the period is:

$$F_0 = \sum_{A_0} F_{i,0}.$$

The expected level annuity benefit payment for the *j*-th individual is given by

$$B_{j,0} = \frac{F_{j,0}}{\ddot{a}_x} = \frac{F_0}{\ddot{a}_x}\frac{F_{j,0}}{F_0} = B_0\left(\frac{F_{j,0}}{F_0}\right)$$

where $B_0 = F_0 / \ddot{a}_x$ is the level annuity benefit payment for the entire group. Thus, it is clear that:

$$B_0 = \sum_{A_0} B_{i,0}.$$

After one period, that is at $t = 1$, the entire group's fund value becomes

$$F_1^* = (F_0 - B_0)(1 + R_1^*)$$

and is used to determine the next annuity payout. For the entire group or cohort, it is

$$B_1^* = \frac{F_1^*}{\ddot{a}_{x+1}}$$

so that the benefit payout rate per unit of fund is equal to

$$\frac{B_1^*}{\sum_{A_1} F_{i,1}^*} = \frac{B_1^*}{F_1^* - \sum_{D_0} F_{i,1}^*}$$

where $\sum_{A_1}$ denotes summation of $F_{i,1}^* = (F_{i,0} - B_{i,0})(1 + R_1^*)$ for those alive at period $t = 1$ and $\sum_{D_0}$ denotes summation over those who died between $[0,1)$. Notice that

$$F_1^* = \sum_{A_1} F_{i,1}^* + \sum_{D_0} F_{i,1}^* = \sum_{A_0} F_{i,1}^* .$$

For any annuitant *j* who is alive at the end of the period, the benefit payout rate can be computed using

$$B_{j,1}^* = \frac{B_1^*}{\sum_{A_1} F_{i,1}^*} \cdot F_{j,1}^* = \frac{F_1^*}{\sum_{A_1} F_{i,1}^*} \cdot \frac{F_{j,1}^*}{\ddot{a}_{x+1}} = \frac{\sum_{A_1} F_{i,1}^* + \sum_{D_0} F_{i,1}^*}{\sum_{A_1} F_{i,1}^*} \cdot \frac{F_{j,1}^*}{\ddot{a}_{x+1}} = \frac{F_{j,1}^* + \left(\sum_{D_0} F_{i,1}^* \Big/ \sum_{A_1} F_{i,1}^* \right) F_{j,1}^*}{\ddot{a}_{x+1}}$$

where the second term in the numerator is an additional benefit to the annuitant derived from a redistribution of the funds available from those who died during the period. One can think of this as a form of "inheritance" derived from those who died in the group. We denote the whole numerator by $\hat{F}_{j,1}^*$. Some algebraic manipulation leads us to a further adjustment formula:

$$B_{j,1}^* = \frac{F_1^*}{\sum_{A_1} F_{i,1}^*} \cdot \frac{F_{j,1}^*}{\ddot{a}_{x+1}} = \frac{F_1^*}{\sum_{A_1} F_{i,1}^*} \cdot \frac{(B_{j,0}\ddot{a}_x - B_{j,0})(1 + R_1^*)}{(\ddot{a}_x - 1)(1 + R) \cdot 1/p_x} = B_{j,0} \cdot \frac{p_x}{\sum_{A_1} F_{i,1}^* \Big/ F_1^*} \cdot \left(\frac{1 + R_1^*}{1 + R} \right).$$

Therefore, the next year's benefit payout rate is calculated by adjusting the previous year's benefit payout rate by a factor due to mortality and another factor due to interest rates, and again we have the pattern of formula (4). We can inductively extend this to time t. The annuity benefit payout rate for an annuitant who survives to time t can be determined using the following adjustment formula:

$$B^*_{j,t} = B^*_{j,t-1} \cdot \frac{p_{x+t-1}}{\sum_{A_t} F^*_{i,t} \Big/ F^*_t} \cdot \left(\frac{1+R^*_t}{1+R} \right).$$

The term

$$\sum_{A_t} F^*_{i,t} \Big/ F^*_t = \left(\sum_{A_t} F^*_{i,t} \right) \Bigg/ \left(\sum_{A_{t-1}} F^*_{i,t} \right)$$

can be interpreted as the realized proportion of the fund surviving from $t-1$ to t. We define

$$p^*_{x+t-1} = \left(\sum_{A_t} F^*_{i,t} \right) \Bigg/ \left(\sum_{A_{t-1}} F^*_{i,t} \right) \tag{5}$$

as the realized survivorship rate of a unit of fund. Thus, we see that this benefit payout formula fits formula (4) where in this case, we have the mortality adjustment

$$MEA_t = \frac{p_{x+t-1}}{\sum_{A_t} F^*_{i,t} \Big/ F^*_t}$$

and the same interest rate adjustment

$$IRA_t = \frac{1+R^*_t}{1+R}.$$

The following simple example, as depicted in Figure 1, shows the effect on the benefit payment when a deviation in mortality occurs in a single period. Here we consider a single individual belonging to the age-60 cohort joining at plan inception whose initial benefit payment is established at $300 per period.

The interest rate used in this and all following examples in the paper is a level 4 percent per annum. The mortality basis used is the U.S. RP-2000 Male Healthy Annuitant, selected for no special reason except that the tables extend to age 120, allowing us to follow retirees over an extended period. These tables form the standard basis for valuing pension plan liabilities in the United States.[3] Life expectancy at age 60 is 21.1 years. After 15 years, 75 percent of the entrants aged 60 are still alive.

Figure 1

Payment Adjustment due to Mortality Deviation in a Single Year

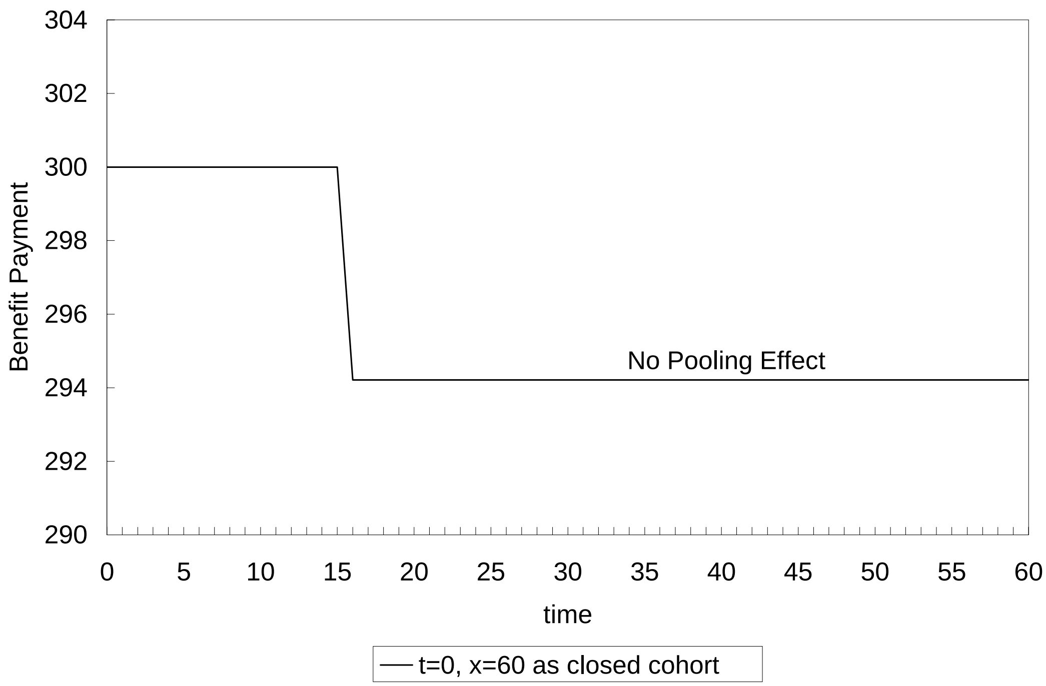


Figure 1: The figure shows the effect of the payment adjustment in the case of deviation in mortality in the period between $t = 15$ and $t = 16$ in the case of a single cohort.

A deviation in the mortality rate at period 15 requires a benefit adjustment from this initial value at time $t = 16$. In particular, a drop in the mortality rate at period 15 by 50 percent causes the benefit payment to drop to $294, approximately a 2 percent drop, and as expected, benefit payment levels off after that.

4. Cohort Analysis

We now introduce cohorts with different annuity factors entering the pool at different points in time. The specifications of (4) now change, of course, so that expectations current at joining are embodied in the new annuity factor. The underlying principle is that the contract offered be actuarially fair at the time it is

[3] See the Society of Actuaries report on the RP-2000 Mortality Tables available at http://www.soa.org/ccm/content/research-publications/experience-studies-tools/the-rp-2000-mortality-tables/.

closed. Otherwise, the funds will either have no new clients or be flooded with takers.

In order to integrate the new entrants with existing members in the fund—and thus exploit risk-pooling—in an actuarially fair manner, two things must happen. First, the benchmark benefit offered must reflect expectations held at the time of joining. Second, the payout paths of all the groups must capture idiosyncratic risk across the two groups in a seamless way.

Four criteria can be formulated to render these operational:

1. If all groups experience mortality equal to expected mortality, payouts should not alter for any group;
2. If groups' expected and actual mortalities differ, payments should all vary in the same proportion;
3. Departures of realized from expected mortality should result in a once-for-all adjustment in all future payments; and
4. Period–by-period fund balance should be preserved.

These requirements seem natural; they are also necessary to maintain actuarially fair offers to new entrants.

There are several time and age dimensions involved when we introduce multiple cohorts. First, there is the age at which an individual enters. We shall denote this by $[x]$ where the bracket symbol $[\cdot]$ is a standard actuarial symbol to indicate differences in mortality pattern due to selection. Second, we have the current period, indicated by time t. Lastly, we have the length of time that has elapsed since joining the plan. This will be indicated by k so that $t-k$ denotes the time at entry measured from plan inception at time 0. Thus, when $t=k$, these are the cohorts who joined at plan inception.

We shall denote by

$$ {}_{[x]}^{k}F_{i,t}^{*} = \left({}_{[x]}^{k-1}\hat{F}_{i,t-1}^{*} - {}_{[x]}^{k-1}B_{i,t-1}^{*} \right)\left(1+R_t^{*}\right) $$

the realized fund value at time t for the *i*-th annuitant belonging to the cohort who entered at age $[x]$, k periods ago. With ${}_{[x]}^{k}\hat{F}_{i,t}^{*}$ we denote the fund value at time t for the *i*-th annuitant including the "inheritance" from those who died between $t-1$ and t for which the value is determined below. At time t, total plan fund therefore is

$$ F_t^{*} = \sum_{k\geq 1}\sum_{x}\sum_{A_{t-1}} {}_{[x]}^{k}F_{i,t}^{*} = \sum_{k\geq 1}\sum_{x}\sum_{A_t} {}_{[x]}^{k}\hat{F}_{i,t}^{*} $$

where A_{t-1} and A_t consist of the individuals alive at time $t-1$ and t, respectively, for the cohort ($[x]$, k). To keep the notation simple, we do not label the cohorts apart from time. Notice that we sum only for those who entered *before* t, not for those who enter at t. In developing the adjusted payment, we consider those exposed to risk in the previous period, but not those who are only entering now, i.e., the new entrants. For new entrants, the payment is determined by

$$ {}_{[x]}^{0}B_{j,t} = {}_{[x]}^{0}F_{j,t} \big/ \ddot{a}_{[x]} \tag{6} $$

with ${}_{[x]}^{0}F_{j,t}$ denoting the amount of fund they invest. There is no right or obligation for these new entrants to participate in any imbalances caused by deviations in the previous period because they were not at that time pool members.

We shall denote by ${}_{[x]}^{k}B_{i,t}^{*}$ the payment at time t for the *i*-th annuitant belonging to the cohort who entered at age $[x]$, k periods ago. At time t, the total benefit payment is

$$ B_t^{*} = \sum_{k\geq 1}\sum_{x}\sum_{A_t} {}_{[x]}^{k}B_{i,t}^{*}, $$

again the summation is only for those who have been members of the plan at time t.

Let us examine our four criteria in mathematical terms. The last two criteria require that

$$ \sum_{k\geq 1}\sum_{x}\sum_{A_t}\left({}_{[x]}^{k}B_{i,t}^{*}\ddot{a}_{[x]+k}\right) = F_t^{*}. \tag{7} $$

Equation (7) balances the fund and the present value of the future payments. Setting the future payments constant restricts the fund balancing response to a disturbance in mortality to a once-for-all adjustment in all future payouts. With ${}_{[x]}^{k}B_{i,t}^{*} = {}_{[x]}^{k-1}B_{i,t-1}^{*} \times MEA_t \times IRA_t$, which implies that MEA_t must be independent of the cohort the member belongs to (criterion 2), on the LHS we get

$$ \sum_{k\geq 1}\sum_{x}\sum_{A_t}\left({}_{[x]}^{k}B_{i,t}^{*}\ddot{a}_{[x]+k}\right) = \sum_{k\geq 1}\sum_{x}\sum_{A_t}\left[{}_{[x]}^{k-1}B_{i,t-1}^{*}\frac{1+R_t^{*}}{1+R}\cdot\left(\ddot{a}_{[x]+k-1}-1\right)\frac{1+R}{p_{[x]+k-1}}\right] $$

$$ = MEA_t\sum_{k\geq 1}\sum_{x}\sum_{A_t}\left[\left({}_{[x]}^{k-1}B_{i,t-1}^{*}\ddot{a}_{[x]+k-1} - {}_{[x]}^{k-1}B_{i,t-1}^{*}\right)\frac{1+R_t^{*}}{p_{[x]+k-1}}\right] $$

with $\left({}_{[x]}^{k-1}B_{i,t-1}^{*}\ddot{a}_{[x]+k-1} - {}_{[x]}^{k-1}B_{i,t-1}^{*}\right)\left(1+R_t^{*}\right) = \left({}_{[x]}^{k-1}\hat{F}_{i,t-1}^{*} - {}_{[x]}^{k-1}B_{i,t-1}^{*}\right)\left(1+R_t^{*}\right) = {}_{[x]}^{k}F_{i,t}^{*}$.

Thus this becomes:

$$\sum_{k\geq 1}\sum_{x}\sum_{A_t}\left({}_{[x]}^{k}B_{i,t}^{*}\,\ddot{a}_{[x]+k}\right)=MEA_t\sum_{k\geq 1}\sum_{x}\sum_{A_t}\left[\left(p_{[x]+k-1}\right)^{-1}\,{}_{[x]}^{k}F_{i,t}^{*}\right]=F_t^{*}.$$

Solving this, the result for our adjustment factor is:

$$MEA_t=\frac{F_t^{*}}{\sum_{k\geq 1}\sum_{x}\left(p_{[x]+k-1}\right)^{-1}\sum_{A_t}{}_{[x]}^{k}F_{i,t}^{*}}. \tag{8}$$

The factor also satisfies our first criterion: if no deviations occur between $t-1$ and t, then

p/p^{*} is simply equal to 1 for each cohort and MEA_t will be equal to unity. It follows that our four criteria of fairness lead to this unique formula for MEA_t.

We transform the adjustment factor a little bit further and use definition (5):

$$\frac{F_t^{*}}{\sum_{k\geq 1}\sum_{x}\left(p_{[x]+k-1}\right)^{-1}\sum_{A_t}{}_{[x]}^{k}F_{i,t}^{*}}=\frac{F_t^{*}}{\sum_{k\geq 1}\sum_{x}\left(p_{[x]+k-1}\right)^{-1}\cdot\sum_{A_t}{}_{[x]}^{k}F_{i,t}^{*}\Big/\sum_{A_{t-1}}{}_{[x]}^{k}F_{i,t}^{*}\cdot\sum_{A_{t-1}}{}_{[x]}^{k}F_{i,t}^{*}}$$

$$=\frac{1}{\sum_{k\geq 1}\sum_{x}\frac{p_{[x]+k-1}^{*}}{p_{[x]+k-1}}\cdot\sum_{A_{t-1}}{}_{[x]}^{k}F_{i,t}^{*}\Big/F_t^{*}}$$

This last term is a harmonic mean of the ratios p/p^{*} of all cohorts.[4] Thus, we have the following weighted harmonic mean of these ratios:

$$HM\left(p/p^{*}\right)=\frac{1}{\sum_{k\geq 1}\sum_{x}\left[\frac{p_{[x]+k-1}^{*}}{p_{[x]+k-1}}\cdot\sum_{A_{t-1}}{}_{[x]}^{k}F_{i,t}^{*}\Big/F_t^{*}\right]}.$$

This is the link between the single-cohort case and the multiple-cohort case: the adjustment factor for the multiple cohorts is the weighted harmonic mean of the individual adjustment factors.

Once again the payment ${}_{[x]}^{k}B_{i,t}^{*}$ at time t follows our general formula (4) with

$${}_{[x]}^{k}B_{i,t}^{*}={}_{[x]}^{k-1}B_{i,t-1}^{*}\cdot\frac{F_t^{*}}{\sum_{k\geq 1}\sum_{x}\left(p_{[x]+k-1}\right)^{-1}\sum_{A_t}{}_{[x]}^{k}F_{i,t}^{*}}\cdot\frac{1+R_t^{*}}{1+R} \tag{9}$$

[4] The harmonic mean is equivalent to taking the arithmetic average of the reciprocals of the ratios and taking the reciprocal of the result.

where the summation $\sum_{x}$ is taken over all entry ages and the summation $\sum_{k\geq 1}$ is taken over all pool members' life durations. As stated earlier, this formula is inclusive of cohorts who entered prior to period t but not those who enter at exactly period t.

Recall that in the previous sections where we only have a single cohort, the total fund available at any time was annuitized among the survivors using a single annuity factor applicable for the cohort. For multiple cohorts, we have multiple annuity factors, but using an average-type annuity factor accounting for the multiplicity of the cohorts will assist us in developing a similar approach to formula (4) as in the previous sections. To illustrate, annuitizing the total available fund, we have the following level annuity benefit payment for all the cohorts:

$$B_t^* = \frac{F_t^*}{avg[\ddot{a}(t)]} = \frac{\sum_{k\geq 1}\sum_{x}\sum_{A_{t-1}} {}_{[x]}^{k}F_{i,t}^*}{avg[\ddot{a}(t)]} \tag{10}$$

where $avg[\ddot{a}(t)]$ is the annuity factor averaged across cohorts.

Notice that

$$B_t^* = \sum_{k\geq 1}\sum_{x}\sum_{A_t} {}_{[x]}^{k}B_{i,t}^* = \sum_{k\geq 1}\sum_{x}\sum_{A_t}\left[{}_{[x]}^{k-1}B_{i,t-1}^* \cdot \frac{F_t^*}{\sum_{k\geq 1}\sum_{x}\left(p_{[x]+t-1}\right)^{-1}\sum_{A_t} {}_{[x]}^{k}F_{i,t}^*} \cdot \frac{1+R_t^*}{1+R}\right]$$

so that we have

$$B_t^* = F_t^* \times \left(\frac{\sum_{k\geq 1}\sum_{x}\sum_{A_t} {}_{[x]}^{k-1}B_{i,t-1}^*}{\sum_{k\geq 1}\sum_{x}\left(p_{[x]+t-1}\right)^{-1}\sum_{A_t} {}_{[x]}^{k}F_{i,t}^*}\right) \times \left(\frac{1+R_t^*}{1+R}\right). \tag{11}$$

The reciprocal of the last two factors, which is equal to

$$\left(\frac{\sum_{k\geq 1}\sum_{x}\left(p_{[x]+t-1}\right)^{-1}\sum_{A_t} {}_{[x]}^{k}F_{i,t}^*}{\sum_{k\geq 1}\sum_{x}\sum_{A_t} {}_{[x]}^{k}B_{i,t-1}^*}\right) \times \left(\frac{1+R}{1+R_t^*}\right) \tag{12}$$

can be interpreted as the average annuity factor $avg[\ddot{a}(t)]$ appearing in equation (10).

In an effort to find a form of $avg[\ddot{a}(t)]$ that resembles individual annuity factors, we show in the appendix that a definition of $avg[\ddot{a}(t)]$ is possible that leads to an approximation of formula (12).

Returning to our simple example, we show in Figure 2 the effect of pooling longevity over several age cohorts when a deviation in mortality occurs in a single period. Again, we consider the single individual belonging to the age-60 cohort joining at plan inception whose initial benefit payment is established at $300 per period. In addition, other cohorts of differing ages were permitted to enter at a later time. Upon entering, these new cohorts start with the same fund endowment as the first one. For purposes of simplifying the illustration, these new cohorts do not encounter any deviation from mortality expectation. As depicted in Figure 2 below, when we pool all the cohorts together, the effect of a drop in mortality for one particular group is less dramatic, as anticipated. In this example, the payment drops by less than 0.5 percent to $298.7, much less than the 2 percent computed in the previous section.

Figure 2

The Effect of Pooling over Several Cohorts

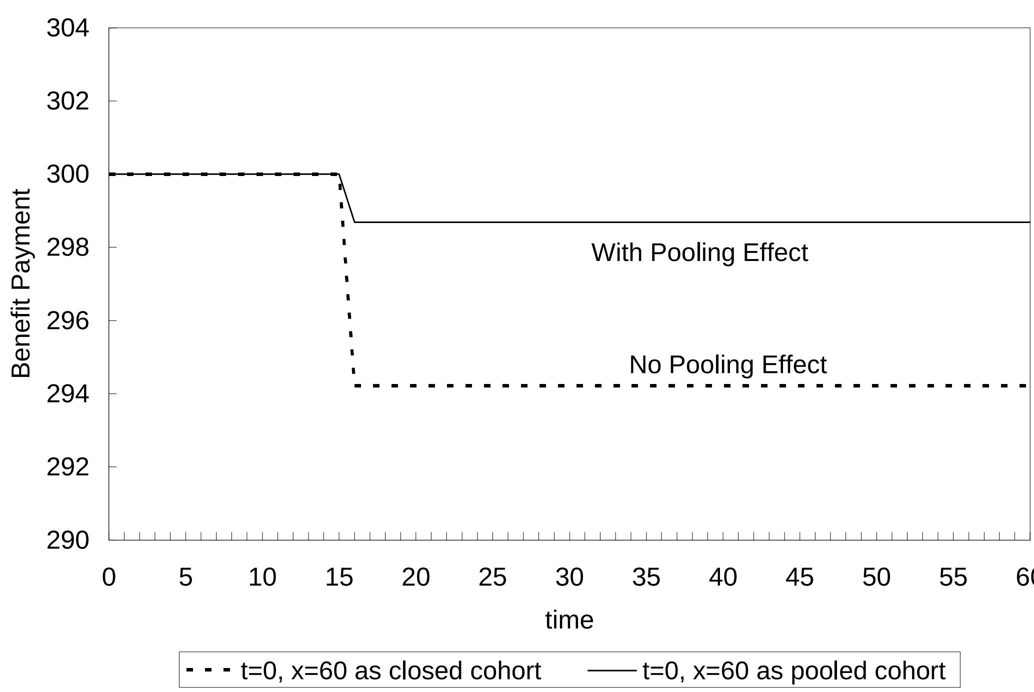


Figure 2: The figure shows the impact of a 50 percent drop in mortality rate in period 15 on the benefit payment from period 16 and onwards, without and with pooling.

5. Fixed versus Evolving Expectations

For a conventional annuity, individual mortality and the interest rate are fixed at the point of entry into the plan. Expected annuity calculations in successive years are therefore based on the mortality and interest expectations at entry. However, in reality, mortality patterns and the interest rate are evolving through time. For a GSA annuity, where annuitants bear systematic longevity risk, changed expectations must be reflected in these annuity and benefit calculations as they emerge.

To fix ideas, let us assume for the moment that new annuity factors become available at some future time t. For any individual entering at this time t, following formula (6), the new annuity factors form the basis for determining the initial benefit payout which we have denoted by ${}_{[x]}^{0}B_{j,t}$. We must also incorporate this new information of future mortality in the computation of the payments for existing members. It is anticipated that an additional adjustment factor is necessary. To determine the appropriate adjustment, we introduce the adjustment factor, CEA_t, for changed expectation of which the base values of p and R and the old annuity factors are superscripted with ^{old} and the base values of p and R and the new annuity factors are superscripted with ^{new}.

For existing members, as well as for new members, the payment made at time t to each member must be equal to the fund allocated to that member at that time divided by the annuity factor. Incorporating the new mortality information, we use $\ddot{a}_{[x]+k}^{new}$ at time t so that

$$ {}_{[x]}^{k}B_{i,t}^{*} = \frac{{}_{[x]}^{k}\hat{F}_{i,t}^{*}}{\ddot{a}_{[x]+k}^{new}}, $$

where ${}_{[x]}^{k}\hat{F}_{i,t}^{*}$ is the fund allocated to an existing member that includes all the additional fund "inherited" from non-surviving members from previous period. It can be shown that

$$ {}_{[x]}^{k}\hat{F}_{i,t}^{*} = {}_{[x]}^{k}F_{i,t}^{*} \cdot 1 \big/ p_{[x]+k-1}^{old} \cdot MEA_t , $$

with ${}_{[x]}^{k}F_{i,t}^{*} = \left({}_{[x]}^{k-1}\hat{F}_{i,t-1}^{*} - {}_{[x]}^{k-1}B_{i,t-1}^{*}\right)\left(1+R_t^{*}\right)$, MEA_t defined as in formula (8), and ${}_{[x]}^{k}F_{i,t}^{*}$ being the fund from time $t-1$ to t without recognizing the increase in the fund over this period due to redistribution of wealth of members dying in that period.

Algebraic manipulation leads us to:

$$
\begin{aligned}
{}_{[x]}^{\;k}B_{i,t}^{*} &= \frac{\left({}_{[x]}^{k-1}\hat{F}_{i,t-1}^{*} - {}_{[x]}^{k-1}B_{i,t-1}^{*}\right)\left(1+R_t^{*}\right)\cdot 1/ p_{[x]+k-1}^{old}\cdot MEA_t}{\ddot{a}_{[x]+k}^{new}} \frac{\ddot{a}_{[x]+k}^{old}}{\ddot{a}_{[x]+k}^{old}} \\
&= \frac{\left({}_{[x]}^{k-1}B_{i,t-1}^{*}\ddot{a}_{[x]+k-1}^{old} - {}_{[x]}^{k-1}B_{i,t-1}^{*}\right)\left(1+R_t^{*}\right)\cdot 1/ p_{[x]+k-1}^{old}\cdot MEA_t}{\left(\ddot{a}_{[x]+k-1}^{old}-1\right)\left(1+R^{old}\right)\cdot 1/ p_{[x]+k-1}^{old}} \frac{\ddot{a}_{[x]+k}^{old}}{\ddot{a}_{[x]+k}^{new}} \\
&= {}_{[x]}^{k-1}B_{i,t-1}^{*}\cdot MEA_t \frac{\left(1+R_t^{*}\right)}{\left(1+R^{old}\right)} \frac{\ddot{a}_{[x]+k}^{old}}{\ddot{a}_{[x]+k}^{new}} \\
&= {}_{[x]}^{k-1}B_{i,t-1}^{*} \times MEA_t \times IRA_t \times CEA_t
\end{aligned}
$$

This may be usefully compared with (4) above. Here, we have an additional factor

$$
CEA_t = \frac{\ddot{a}_{[x]+k+1}^{old}}{\ddot{a}_{[x]+k+1}^{new}}
$$

to account for the new mortality information available at time t.

Assuming this same information of future mortality and interest rates carries to subsequent periods, we continue adapting $\ddot{a}_{[x]+k+1}^{new}$ and start with:

$$
\begin{aligned}
{}_{[x]}^{k+1}B_{i,t+1}^{*} &= \frac{\left({}_{[x]}^{\;k}\hat{F}_{i,t}^{*} - {}_{[x]}^{\;k}B_{i,t}^{*}\right)\left(1+R_{t+1}^{*}\right)\cdot 1/ p_{[x]+k}^{old}\cdot MEA_{t+1}}{\ddot{a}_{[x]+k+1}^{new}} \\
&= {}_{[x]}^{k-1}B_{i,t-1}^{*} \frac{\left(\ddot{a}_{[x]+k}^{new} - {}_{[x]}^{\;k}B_{i,t}^{*}\right)\left(1+R_{t+1}^{*}\right)}{\left(\ddot{a}_{[x]+k}^{new}-1\right)\left(1+R^{new}\right)}\cdot MEA_{t+1} \\
&= {}_{[x]}^{\;k}B_{i,t}^{*} \times MEA_{t+1} \times IRA_{t+1}
\end{aligned}
$$

which brings us back to the adjustment formula in the known form. This demonstrates that the adjustment factor CEA_t is necessarily applied only once at the period the mortality basis changes. We illustrate the impact of excluding the new knowledge in the benefit calculation of existing members in Figure 3, which displays a comparison of the annuity values between old and changed expectations.

Figure 3
Fixed versus Evolving Expectations

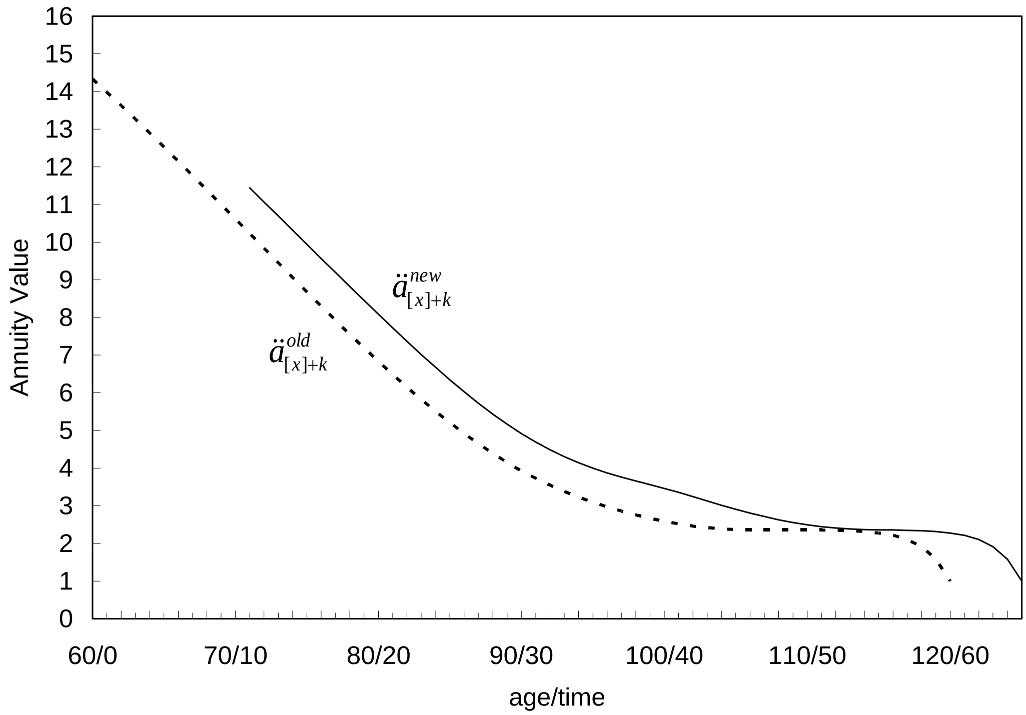


Figure 3: This figure displays a comparison of the "old" and the "new" annuity values over age/time for a particular entry age x = 60 when mortality expectations change at a certain point in time t = 11.

The "old" mortality basis is the RP-2000 mortality table for Male Healthy Annuitant. In year eleven, we introduce the "new" mortality basis for a male individual, which we hypothetically assume to be equal to the RP-2000 mortality table for a Female Healthy Annuitant, but further extended by five years, prolonging therefore the ultimate lifetime to age 125. All the annuity values are based on a discount rate of 4 percent, which for simplicity we assume constant. As expected, the "new" annuity values based on the "new" mortality basis are larger and extend five years into the future. Returning to our previous example, we have a single cohort of individuals aged 60 with entry at plan inception at t=0. We additionally assume that we do not have any deviation of mortality from expected, i.e., in each year, the realized mortality in the cohort is equal to the expectation of mortality at that point in time. If we do not integrate the new knowledge of future mortality in year 11, the mortality expectation adjustment factor p_t / p_t^* then equals p_t^{old} / p_t^{new} and it will be different from period 1 for the following ages as shown in Figure 3. From year 55, the new and the old expectation of mortality is equal, hence the adjustment factor becomes 1 again. Also notice that the adjustment factor drops to zero when the original final age of 120 is reached at time $t = 60$.

Figure 4

Ratio of "Old" and "New" Expected Survival Rates

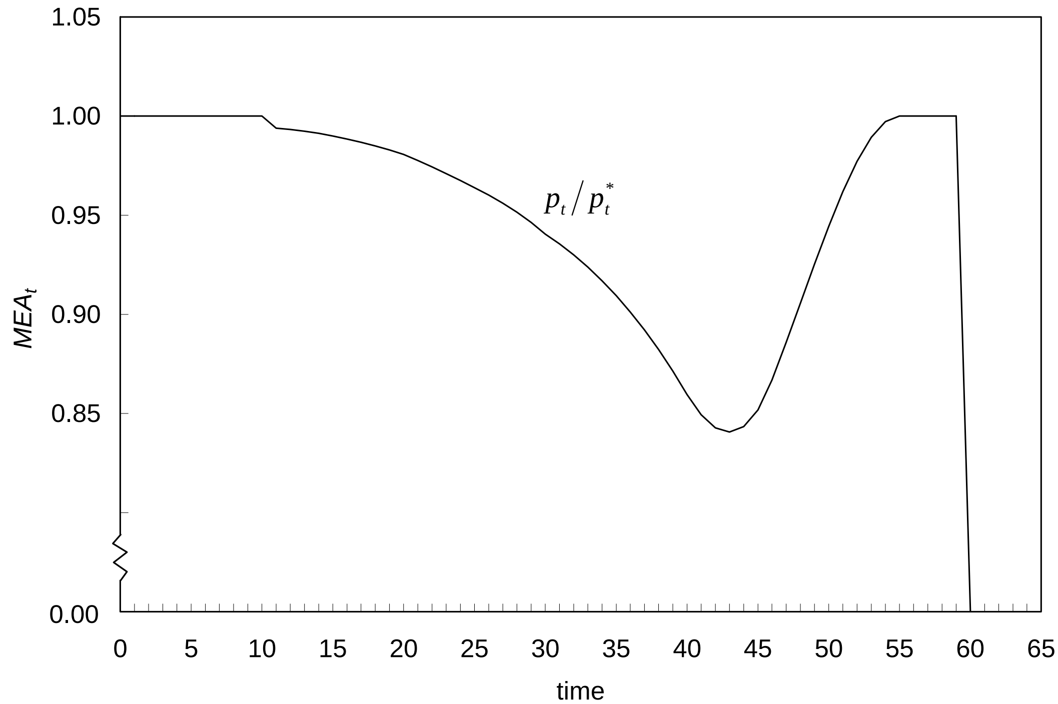


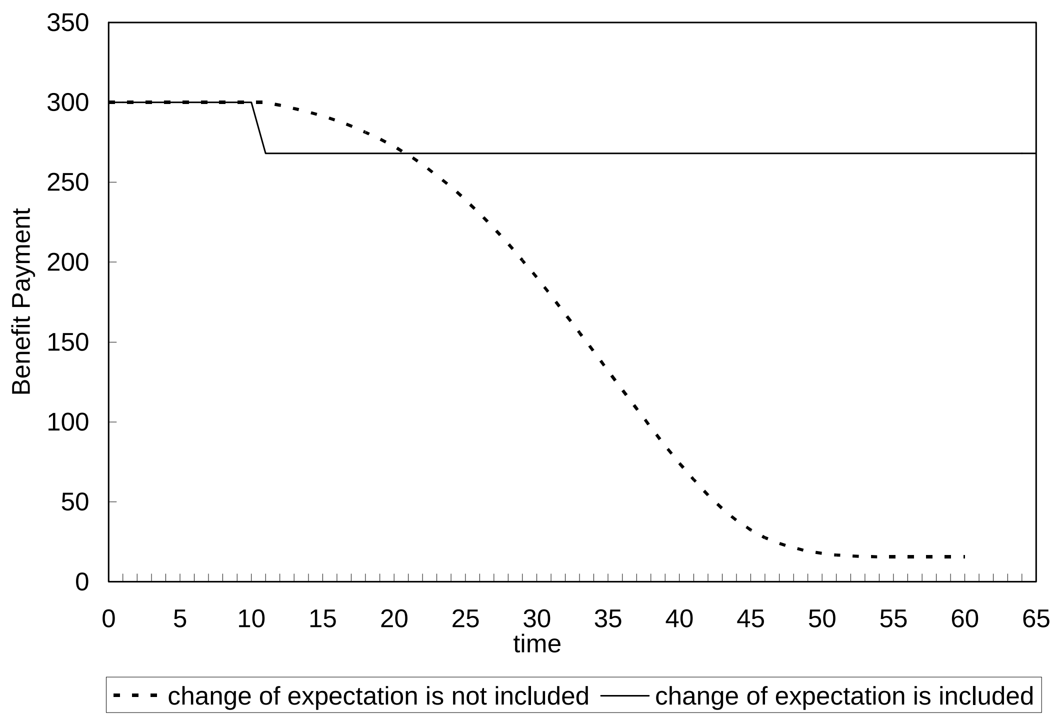
Figure 4: The graph shows the deviation between "old" and "new" expectation for survival probabilities over time.

As before, benefit payments start at $300 per period. As we do not have deviation prior to year 11, these payments remain constant for that period. If we do not adjust the payments to account for new expectations in mortality at the start of year 11, the benefit payment declines from year 12 as reflected in the adjustment factor in Figure 3. After year 60, the payment drops to zero, because according to the old mortality expectation, all individuals would have been presumed dead. When the "new" basis of mortality is reflected immediately at the time when the information becomes readily available, this causes an adjustment at time $t = 11$ of $\ddot{a}^{old}_{[60]+10} \big/ \ddot{a}^{new}_{[60]+10} = 0.893$ or roughly indicating an 11 percent decrease, dropping the payment to $268. In subsequent periods, we again observe a constant pattern because we have assumed no further deviation between realized and new expected mortality. Furthermore, notice that the payments continue until the new assumed final age of 125.

This is one very strong argument for integrating all new knowledge as it becomes available—the new knowledge is immediately crystallized in all subsequent payments. Although there are noticeable differences in the periodic cash flow pattern, as displayed in Figure 5, it is expected both patterns have an equal actuarial present value of $3,066.96 at time $t = 11$, assuming of course the interest rate of 4 percent and the new mortality expectation to discount the payments.

Figure 5

Fixed versus Evolving Expectations



[0]

Figure 5: This figure displays a comparison of the level of benefit payments between a "change" and a "no-change" in expectations.

In the case of a "change" in expectations, zero deviation to the (new) expected mortality is assumed; hence the payment is constant after the adjustment in period 11 while in the case of a "no-change," the payment shows a steady decline due to the difference in the "old" and "new" expectations.

6. Numerical Illustration

To illustrate the ideas developed in the previous sections with a more realistic example, we present here a numerical representation of a GSA plan. As in earlier examples, the RP-2000 Mortality Tables form the basis of our mortality assumptions. The expected investment earnings rate has similarly been assumed to be a constant rate of 4 percent. While the realization of the investment returns can have a dramatic impact on the values of the annuities as well as the resulting benefit payout rates, the realized investment return has also been assumed constant. We do not attempt to measure the impact of deviations from returns in this illustration. While deviations in interest rates are likely to have a greater magnitude of impact than the financial risk resulting from longevity variation, we focus here on the financial consequences of longevity risk. As shown in the development of the model, every individual encounters the same interest rate risk, so there is no gain that can be made from pooling in this dimension. Financing the risks of longevity through "pooling" is the primary focus of this paper.

The six different cohorts that enter our GSA plan over time are depicted in Table 1. Each cohort starts with the same initial fund; notice that the choice of the size of the fund is immaterial to the results as long as each cohort is endowed with the same initial wealth.

Table 1

Description of Cohorts in the Example

Cohort	Time at Entry from Plan Inception *t*	Age at Entry *x*	Starting Payment of Individual	Beginning Fund of Individual
1	0	75	350	3,036
2	0	60	400	5,734
3	10	60	450	6,451
4	20	60	500	7,167
5	20	85	550	2,862
6	30	60	600	8,601

The payment ${}_{[x]}^{0}B_{i,t}^{*}$ at entry for a representative individual in each cohort is distinguished according to increments of $50 starting from $350 up to $600. This choice has been arbitrarily made for the only reason to get less overlapping in the graphs showing the benefit payment pattern over time.

The deviations in mortality rates have been modeled to capture future random variation. While this is not a straightforward process, the choice has been made to keep the illustration again simple. More sophisticated models of uncertainty in future mortality trends can be found, for example, in Olivieri and Pitacco (2003) and Haberman and Vigna (2002). Even though the results may differ when using a different statistical assumption, the important feature we want to show—the decrease in volatility achieved by pooling the cohorts—should not be affected in principle. One obvious feature that should be captured is the greater variability of mortality with increasing age. The deviations within a cohort must increase with age due to the decreasing number of survivors or remaining lives. This can be empirically observed by comparing crude mortality data with graduated data in the process of building a mortality table, where the fitness in the older ages is generally poor. After some investigation of Australian mortality tables, we find that a reasonable bound for the deviation even in very high ages appears to be in the neighborhood of 20 percent. These empirical and intuitive observations led us to model the realization of mortality rates over time using the following formula:

$$q_x^* = q_x \cdot \left\{ \frac{x}{100} \cdot \left[U(0,1) \cdot 0.3 - 0.15 \right] + 1 \right\}, \tag{13}$$

where $U(0,1)$ is a uniform random variable which can easily be generated even in a spreadsheet. From this formula, it is clear that we assume the randomness in mortality to increase linearly with age.

Figure 6

Model of a GSA Plan with Six Different Cohorts, No Pooling

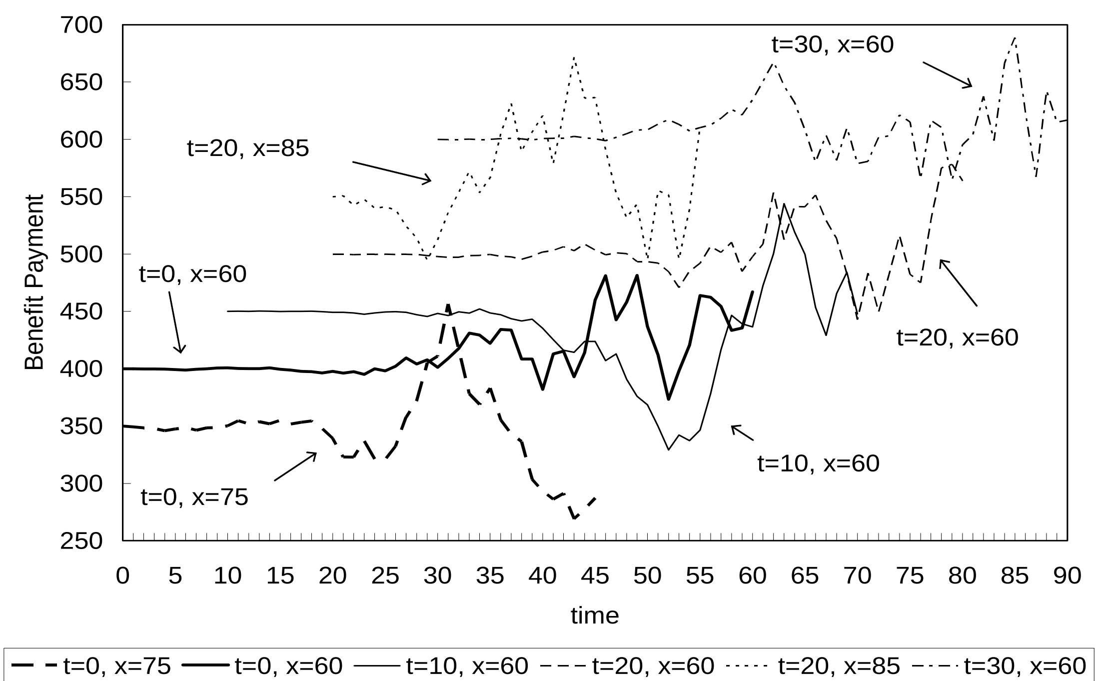


Figure 6: This figure shows that there is strong volatility in the benefit payments where we have no pooling over the age cohorts. Volatility increases clearly with age due to the randomness assumptions.

Figures 6 and 7 graphically depict the variations in payout with and without pooling. The smoothing effect of pooling can easily be discerned in Figure 7, and especially between t = 30, when the last of the sequence of cohorts joins the pool, to t = 55, when the first cohort is fully deceased. Even with pooling, of course, the last cohort to enter faces volatility in payouts in old age, as its surviving members thin out, and there is no younger cohort to cushion unexpected deviations from expectation.

An individual at retirement has the choice of joining one of these GSA plans or buying an ordinary annuity directly from an insurer. To make such an assessment, we would have to simulate the pattern of actual-to-expected mortality that would possibly emerge over time. According to our simulation formula in (13), we have the ratio of actual-to-expected survival probabilities

$$\frac{p_x^*}{p_x} = \frac{1-q_x^*}{1-q_x} = \left(1+0.15\frac{x}{100}\cdot\frac{q_x}{1-q_x}\right) - 0.3\frac{x}{100}\cdot\frac{q_x}{1-q_x}U\left(0,1\right), \tag{14}$$

and it is straightforward to show that this ratio is uniformly distributed on the interval:

$$\left(1-0.15\frac{x}{100}\cdot\frac{q_x}{1-q_x}, 1+0.15\frac{x}{100}\cdot\frac{q_x}{1-q_x}\right).$$

Figure 7

The Effect of Pooling within the Modeled GSA Plan

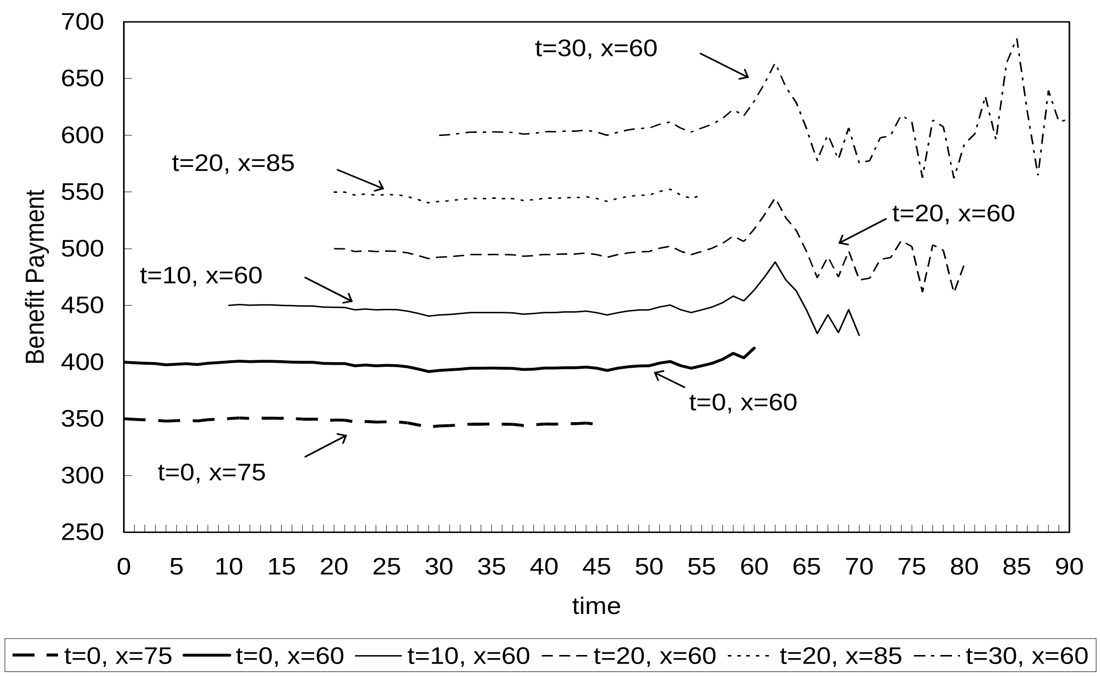


Figure 7: The graph displays that where we pool, some of the deviations cancel out, thus the payments are less volatile. Since from time 30 no new entries occur, volatility increases again while the number of those alive slowly decreases.

Denoting the ratio in (14) by $Y_x = p_x^* / p_x$, the actual benefits that will emerge over time as shown in equation (3), ignoring the effect of interest rate, will be

$$B_t^* = B_{t-1}^* \cdot \frac{p_{x+t-1}}{p_{x+t-1}^*} = B_0 \cdot \prod_{j=1}^{t} Y_{x+j-1}^{-1} .$$

Thus, we can then compute the probability that an individual will have a higher income from a GSA plan than from an ordinary annuity. Insurance companies typically assess premium loading to cover for risk and profits, which reduces the benefit payout to the individual. Assume that this reduction is denoted by λ with $0 \le \lambda < 1$. Then the required probability can be expressed as:

$$P\left(B_t^* > (1-\lambda) B_0\right) = P\left(B_0 \cdot \prod_{j=1}^{t} Y_{x+j-1}^{-1} > (1-\lambda) B_0\right) = P\left(\prod_{j=1}^{t} Y_{x+j-1} < (1-\lambda)^{-1}\right). \qquad (15)$$

Figure 8 displays this probability pattern for various levels of λ. Here we also assume an individual is currently age 65. These results are also based on 1,000 yearly simulation runs. In the case where there is no premium loading, that is $\lambda = 0$, this probability stays as expected, at the approximate level of 0.5 for each year. We observe that for higher levels of premium loading or benefit reduction λ, the probability that benefits from GSA will be higher than ordinary annuities quickly approaches to one especially in the early years.

Figure 8 may also be interpreted to provide guidance on the probability that GSA payments will fall below a given proportion of the first-year payout through its lifetime. For the stochastic process we have adopted, there is almost no chance that payouts will fall below 90 percent of the first-year value before the age of 90. This can be seen by tracking the probability that benefits from the GSA are higher than an ordinary annuity with a 10 percent loading. While systematic longevity risk is little understood, this simulation suggests that the risk of the force of systematic longevity bias dramatically reducing GSA payouts is low.

Figure 8

The Probability Benefits from GSA are Higher than Ordinary Annuities

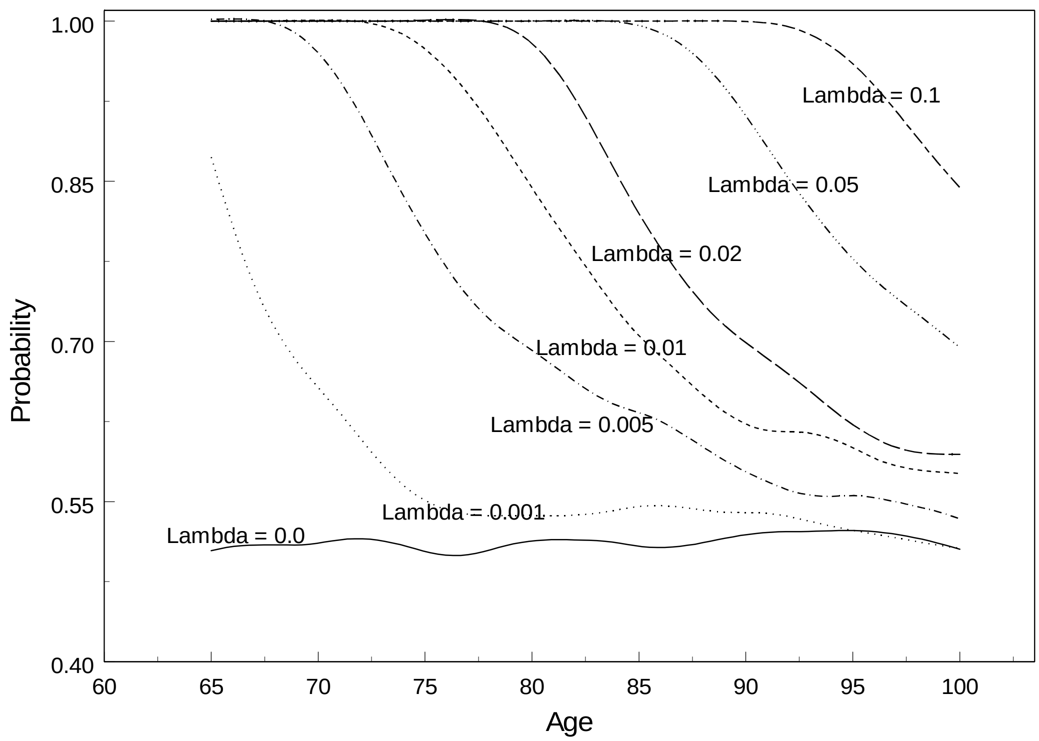


Figure 8: The graph displays the probability that benefit payouts from a GSA plan will exceed those from an Ordinary Annuity issued by an insurance company, reflecting the fact that insurers generally assess a premium loading that reduces the benefit amount. The lambda variable reflects the reduction in the benefit payouts as a result of the loading.

7. Concluding Comments

Longevity risk is becoming an increasingly important issue for retirees, as changing health and lifestyle lead to longer life expectancies. To mitigate the financial risk associated with improvements in longevity, a natural response for the individual is to annuitize (Brown, et al. (2001) and Auerbach and Herrmann (2002)). This paper analyzes the payout implications of pooling longevity risk through GSA, an arrangement in which the annuitants bear their pool's systematic risk but share idiosyncratic risk; we determine specifications of the stream of benefit payments that would emerge in a GSA plan, assuming actuarial fairness, and provide adjustment formulae for payout streams under a range of assumptions. The resulting benefit payment at any given period is shown to be equal to the previous period's benefit payment multiplied by a mortality experience and interest-rate experience adjustment. These adjustments account for deviations of these experiences from expectations. By extending this analysis to several cohorts pooled into a single annuity fund, any variation resulting from sharing the idiosyncratic risk can be reduced. Regulatory and marketing obstacles remain for the practical implications of a GSA, along with issues related to adverse selection; we plan to address some of these in subsequent research.

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Appendix A

To develop a recursive relation between $avg[\ddot{a}(t-1)]$ and $avg[\ddot{a}(t)]$, first define

$$avg[\ddot{a}(t)] = \sum_{l=0}^{\infty} v^l \cdot \left[\prod_{n=0}^{l-1} avg^t(p_{t+n}) \right] \tag{16}$$

with

$$avg^t(p_{t+s}) = \left[\sum_x \sum_k \left(p_{[x]+k+s}\right)^{-1} \cdot \left(\sum_{i \in A_{t-1}} {}_{[x]}^{k}F_{i,t}^{*} \Big/ F_t^{*} \right) \right]^{-1} \quad \forall s \in \mathrm{I}\,. \tag{17}$$

Equation (17) gives an average survivorship rate based on a weighted average of the reciprocals of each cohort's survivorship rate (harmonic mean again). The weights used in (17) can be denoted as

$${}_{[x]}^{k}w_t = \sum_{i \in A_{t-1}} {}_{[x]}^{k}F_{i,t}^{*} \Big/ F_t^{*}$$

and these weights are to be determined at time t. From (16), it follows therefore that

$$avg[\ddot{a}(t-1)] = \sum_{l=0}^{\infty} v^l \cdot \left[\prod_{n=0}^{l-1} avg^{t-1}(p_{t-1+n}) \right]$$

where similar to (17),

$$avg^{t-1}(p_{t-1+s}) = \left[\sum_x \sum_k \left(p_{[x]+k-1+s}\right)^{-1} \cdot {}_{[x]}^{k-1}w_{t-1} \right]^{-1} \tag{18}$$

with weights determined at $t-1$.

Furthermore, we observe that

$$avg[\ddot{a}(t-1)] = \sum_{l=0}^{\infty} v^l \left[\prod_{n=0}^{l-1} avg^{t-1}(p_{t-1+n}) \right] = 1 + v \cdot avg^{t-1}(p_{t-1}) \sum_{l=0}^{\infty} v^l \left[\prod_{n=0}^{l-1} avg^{t-1}(p_{t+n}) \right] \tag{19}$$

Because of the differences in the weights used in definition (17) and (18), the above formula (19) can only be approximately expressed as

$$avg[\ddot{a}(t-1)] \approx 1 + v \cdot avg^t(p_{t-1}) \cdot avg[\ddot{a}(t)]. \tag{20}$$

Using definitions (16) and approximation (20) in formula (10), we can show that we get approximately formula (11). To prove this, first notice that

$$
\begin{aligned}
B_t^* &= \frac{F_t^*}{avg\left[\ddot{a}(t)\right]} = \frac{\left(F_{t-1}^* - B_{t-1}^*\right)\left(1 + R_t^*\right)}{avg\left[\ddot{a}(t)\right]} \\
&= \frac{\left(B_{t-1}^* \cdot avg\left[\ddot{a}(t-1)\right] - B_{t-1}^*\right)\left(1 + R_t^*\right)}{avg\left[\ddot{a}(t)\right]} \\
&= \frac{B_{t-1}^*\left(avg\left[\ddot{a}(t-1)\right] - 1\right)\left(1 + R_t^*\right)}{avg\left[\ddot{a}(t)\right]}
\end{aligned}
$$

and now using the approximation in (20) and assuming weights to be according to

${}_{[x]}^{k}w_t = \sum_{i \in A_{t-1}} {}_{[x]}^{k}F_{i,t}^* \Big/ F_t^*$, we have

$$
B_t^* \approx \frac{B_{t-1}^* \cdot \left\{avg\left[\ddot{a}(t-1)\right] - 1\right\}}{\left\{avg\left[\ddot{a}(t-1)\right] - 1\right\} \cdot \left[1 \big/ avg^t\left(p_{t-1}\right)\right]} \cdot \left(\frac{1 + R_t^*}{1 + R}\right).
$$

Using definition (17) with $s = -1$ gives us

$$
B_t^* \approx F_t^* \cdot \left(\frac{\sum_k \sum_x \sum_{A_{t-1}} {}_{[x]}^{k}B_{i,t-1}^*}{\sum_k \sum_x \left(p_{[x]+k-1}\right)^{-1} \sum_{A_t} {}_{[x]}^{k}F_{i,t}^*}\right) \cdot \left(\frac{1 + R_t^*}{1 + R}\right), \tag{21}
$$

which clearly approximates equation (9). The primary difference between formulas (9) and (21) above lies in the summation of the annual payments over A_{t-1} instead of A_t in the numerator.◎

Appendix B

In this appendix, we list some of the standard actuarial symbols and other algebraic symbols used in this paper for convenience to the reader. They are listed in the order they appear in the paper.

Symbol	Interpretation
ℓ_x	Number of annuitants left in the pool who are aged x.
v	Present value of one unit payable one year from now: $v = 1/(1+R)$.
$\ddot{a}_x$	Present value of a life annuity issued to individual aged x of one unit per year, payable annually at the beginning of each year provided the annuitant survives.
p_x	Probability that a person aged x will survive one year from now: $p_x = \ell_{x+1} / \ell_x$.
A_t	An index referring to the surviving individuals at time t.
B_t^*	Annual benefit payout made to surviving individuals at time t. The symbol * is used to indicate the actual and is omitted when $t = 0$.
F_t^*	Fund balance in the GSA pool at time t.
$B_{j,t}^*$	Annual benefit payout made to the surviving individual j at time t; each individual may have contributed differently.
$F_{j,t}^*$	Fund balance of individual j at time t.
$[x]$	The entry (or issue) age where the bracket symbol [.] indicates differences in mortality pattern due to selection.
${}_{[x]}^{k}B_{i,t}^*$	Annual benefit payout made to the i-th annuitant belonging to the cohort who entered at age $[x]$, k periods ago.
${}_{[x]}^{k}F_{i,t}^*$	Fund balance of the i-th annuitant belonging to the cohort who entered at age $[x]$, k periods ago.